Outline

**Block 1 (9:00 - 10:30)**
- Foundations of SPH
- Governing equations
- Time integration
- **Example:** Our first SPH solver
- Neighborhood Search

**Block 2 (11:00 - 12:30)**
- Enforcing incompressibility
  - State equation solvers
  - Implicit pressure solvers
- Boundary Handling
  - Particle-based methods
  - Implicit approaches

**Lunch break (60min)**

**Block 3 (13:30 - 15:00)**
- Multiphase fluids
- Highly-viscous fluids
- Vorticity and turbulent fluids
- **Demo:** Splash

**Block 4 (15:30 - 17:00)**
- Deformable solids
- Rigid body simulation
  - Dynamics and coupling
- Data-driven/ML techniques
- Summary and conclusion

**Coffee break (30min)
Smoothed Particle Hydrodynamics
Techniques for the Physics Based Simulation of Fluids and Solids

Part 3
Multiphase Fluids

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Motivation

Fluid Interfaces

Gissler et al. 2019

Complex mixing phenomena

Yang et al. 2015
My First Multi-fluid SPH Solver

- Particles carry attributes individually
  - Mass, rest density
  - Concentration, temperature, viscosity, ...
- Two fluids a and b, with \( \frac{m_a}{\rho_a} = \frac{m_b}{\rho_b} \)
- Buoyancy emerges from individual rest densities
My First Multi-fluid SPH Solver

Switching densities

Boiling

Lavalamp

Lenaerts & Dutre 2009

Müller et al. 2005

Müller et al. 2005

Switching densities

Boiling

Lavalamp
High Density Ratios

\[ \rho_0 = 100 \]

\[ \rho_0 = 1000 \]

\[ \rho_0 \]
High Density Ratios

Solenthaler & Pajarola 2008
Interface Discontinuities

- Standard SPH (SESPH)
  - Cannot handle discontinuities at interfaces
  - Results in spurious and unphysical interface tension
  - Large density differences lead to instability problems

- Adapted SPH
  - Capture density discontinuities across interfaces
  - Stable simulations despite high density ratios
  - We need full control over behavior
Interface Discontinuities

- Problems near interfaces where rest densities and masses vary
- Falsified smoothed quantities

\[ \rho_i = \sum_j m_j W_{ij} \]
Interface Discontinuities

- Problems near interfaces where rest densities and masses vary
- Falsified smoothed quantities

\[ p_i = k_1 \left( \frac{\rho_i}{\rho_0} \right)^{k_2} - 1 \]

where \( \rho_0 \) is the rest density, \( \rho_i \) is the density at a particle, and \( k_1 \) and \( k_2 \) are constants. The pressure deviation causes a density change that can be represented as a quotient of the density deviation and the rest density.

![Diagram showing fluid densities and pressure computations](image-url)
Adapted Density and Pressure

- Use number density \( \delta_i = \sum_j W_{ij} \)
- Adapted density of particle \( i \) given by \( \tilde{\rho}_i = m_i \delta_i \)
- Pressure computation using adapted density \( \tilde{p}_i = k_1 \left( \frac{\tilde{\rho}_i}{\rho^0} \right)^{k_2} - 1 \)

\[ \begin{align*}
\text{Fluid 1} & : \rho_n = 1000 \\
\text{Fluid 2} & : \rho_n = 100
\end{align*} \]

\[ \begin{align*}
\rho & = \text{density} \\
\tilde{\rho} & = \text{adapted density} \\
\rho_{\text{desired}} & = \text{desired density} \\
\rho & = \text{density}
\end{align*} \]
Adapted Forces

- Derive adapted forces
- Substitute adapted density and pressure into the NS pressure term
  \[ F^p_i = -\frac{\nabla \tilde{p}}{\delta} \]
- Apply SPH derivation to get adapted pressure force
  \[ F^p_i = -\sum_j \left( \frac{\tilde{p}_j}{\delta_j^2} + \frac{\tilde{p}_i}{\delta_i^2} \right) \nabla W_{ij} \]
- Similarly derivation of viscosity force
  \[ F^v_i = \frac{1}{\delta_i} \sum_j \frac{\mu_i + \mu_j}{2} \frac{1}{\delta_j} (v_j - v_i) \nabla^2 W_{ij} \]
- For a single phase fluid equations are identical to SESPH
- For multi-fluid simulations interface problems are eliminated
- No performance overhead
- Extended with incompressibility condition [Akinci et al. 12, Gissler et al. 19]
Adapted SPH - Results

Solenthaler & Pajarola 2008
Diffusion Effects

- Diffusion equation \( \frac{\partial C}{\partial t} = \alpha \nabla^2 C \)

- SPH equation \( \frac{\partial C_i}{\partial t} = \alpha \sum_j m_j \frac{C_j - C_i}{\rho_j} \nabla^2 W_{ij} \)

Color diffusion

Temperature diffusion (and phase changes)

Müller et al. 2005

Lenaerts & Dutre 2009

Keiser et al. 2005
Complex Mixing Effects

• Previous work
  – Mixture is only caused by diffusion effects
  – Different phases move at the same bulk velocity as the mixture

• SPH based mixture model [Ren et al. 2014]
  – Mixing and unmixing due to (relative) flow motion and force distribution
  – Dynamics of multi-fluid flow captured using mixture model
  – Spatial distribution of phases modeled using volume fraction (similar to [Müller et al. 05])
  – Drift velocities: Phase velocities relative to mixture average
Mixture Model

- **Phase:**
  - Volume fraction \(\alpha_k\), \(\sum_k \alpha_k = 1, \alpha_k \geq 0\).
  - Phase velocity \(v_k\)

- **Mixture:**
  - Mixture density \((f(\alpha_k))\)
  - Mixture velocity \(v_m\)

- Continuity and momentum equations of the phases and mixture

  - The nonuniform distribution of velocity fields will lead to changes in the volume fraction of each phase
  - The drift velocities play a key role in this interaction mechanism
Mixture Model

- Continuity equation of the mixture model
  \[ \frac{D\rho_m}{Dt} = \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}_m) = 0 \]
  - \( \rho_m \) mixture density
  - \( \rho_m = \sum_k \alpha_k \rho_k \)
  - \( \alpha_k \) volume fraction of phase
  - \( \mathbf{v}_m \) mixture velocity (avg over all phases)
  \[ \mathbf{v}_m = \frac{1}{\rho_m \sum_k \alpha_k \rho_k} \mathbf{v}_m \]

- Momentum equation for the mixture
  \[ \frac{D(\rho_m, \mathbf{v}_m)}{Dt} = -\nabla p + \nabla \cdot (\mathbf{\tau}_m + \mathbf{\tau}_{Dm}) + \rho_m \mathbf{g} \]
  - \( \mathbf{\tau}_m \) viscous stress tensor of the mixture
  - \( \mathbf{\tau}_{Dm} \) diffusion tensor of the mixture (convective momentum transfer between phases)

- The nonuniform distribution of velocity fields will lead to changes in the volume fraction of each phase
- The drift velocities play a key role in this interaction mechanism
Algorithm

3 loops over all particles:

1. Compute density and pressure with SPH

2. Compute drift velocity of each phase / particle
   
   Analytical expression of drift velocity, three terms defining
   - Slip velocity due to body forces
   - Pressure effects that cause fluid phases to move from high to low pressure regions
   - Brownian diffusion term representing phase drifting from high to low concentration

   Update diffusion tensor, advect volume fraction
   (using drift velocity)

3. Compute total force, advect particle
Immiscible and Miscible Liquids

- Immiscible
- Miscible, diffusion disabled
- Miscible, diffusion enabled
- Red / green miscible, immiscible with blue

Ren et al. 2014
More Results

Ren et al. 2014
Limitations and Extensions

- [Ren et al. 14] Uses WCSPH; a divergence-free velocity field cannot be directly integrated since neither the mixture nor phase velocities are zero, even if the material is incompressible.


- [Yan et al. 16] Extension to fluid-solid interaction -> dissolution of solids, flows in porous media, interaction with elastics.