Smoothed Particle Hydrodynamics Techniques for the Physics Based Simulation of Fluids and Solids

VISCOSITY

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Motivation
Viscous Force
Explicit Viscosity
Implicit Viscosity
Results
Outline

- Motivation
- Viscous Force
- Explicit Viscosity
- Implicit Viscosity
- Results
Classical Newtonian Fluid Model

Linear momentum equation:

\[ \rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} + \mathbf{f} \]

- Density \( \rho \)
- Velocity \( \mathbf{v} \)
- Stress tensor \( \mathbf{T} \)
- Force density \( \mathbf{f} \)

Newtonian constitutive model:

\[
\mathbf{T} = -p \mathbb{1} + 2\mu \mathbf{E} \\
\mathbf{E} = \frac{1}{2} \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right)
\]

- Strain rate tensor \( \mathbf{E} \)
- Pressure \( p \)
- Identity \( \mathbb{1} \)
- Dynamic viscosity \( \mu \)

Navier-Stokes equations:

\[ \rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f} \]
Viscous Force

- Viscous force for incompressible fluids:
  \[ f_{\text{visco}} = \mu \nabla^2 \mathbf{v} \]

- Recent SPH solvers either compute the divergence of the strain rate \( \mathbf{E} \) or directly determine the Laplacian of \( \mathbf{v} \).

- Note that strain rate based approaches must enforce a divergence-free velocity field to avoid undesired bulk viscosity.
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Explicit Viscosity

- Standard SPH discretization of this Laplacian:

\[ \nabla^2 v_i = \sum_j \frac{m_j}{\rho_j} v_j \nabla^2 W_{ij} \]

- Disadvantages:
  - Sensitive to particle disorder [Mon05, Pri12]
  - The Laplacian of the kernel changes its sign inside the support radius
Explicit Viscosity

- Alternative: determine one derivative using SPH and the second one using finite differences.

\[ \nabla^2 v_i \approx 2(d + 2) \sum_j \frac{m_j}{\rho_j} \frac{v_{ij} \cdot x_{ij}}{||x_{ij}||^2 + 0.01 h^2} \nabla W_{ij} \]

\[ x_{ij} = x_i - x_j, \; v_{ij} = v_i - v_j, \; d = \text{dimension} \]

- Advantages:
  - Galilean invariant
  - vanishes for rigid body rotation
  - conserves linear and angular momentum
Core idea of XSPH: reduce the particle disorder by smoothing the velocity field:

\[ \hat{v}_i = v_i + \alpha \sum_j \frac{m_j}{\rho_j} (v_j - v_i) W_{i,j} \]

- XSPH can also be used as artificial viscosity model.
- Advantage: The second derivative is not needed.
- Disadvantage: \( \alpha \) is not physically meaningful.
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Takahashi et al. 2015

- Compute the viscous force as divergence of the strain rate $\mathbf{E}$:

$$f_{\text{visco}} = \mu \nabla \cdot \left( \nabla \mathbf{v}_i + (\nabla \mathbf{v}_i)^T \right) = \sum_j m_j \left( \frac{2E_i}{\rho_i^2} + \frac{2E_j}{\rho_j^2} \right) \nabla W_{ij}$$

- Implicit integration scheme

$$\mathbf{v}(t + \Delta t) = \mathbf{v}^* + \frac{\Delta t}{\rho} f_{\text{visco}}(t + \Delta t)$$

- Solve linear system using the conjugate gradient method (CG).
Decompose velocity gradient: \( \nabla v = R + V + S \)

Reduce shear rate by user-defined factor \( 0 \leq \xi \leq 1 \):

\[
\nabla v^{\text{target}} = R + V + \xi S
\]

Reconstruct velocity field by solving linear system with CG:

\[
v_i(t + \Delta t) = \frac{1}{\rho_i} \sum_j m_j \left( v_j(t + \Delta t) + \frac{\nabla v_{i}^{\text{target}} + \nabla v_{j}^{\text{target}}}{2} x_{ij} \right) W_{ij}
\]
Define velocity constraint for each particle with user-defined factor:

\[ C_i(v) = E_i - \gamma E_i = 0, \quad 0 \leq \gamma \leq 1 \]

The constraint is a 6D function due to the symmetric strain tensor.

Solve linear system for corresponding Lagrange multipliers.
The introduced methods are based on the strain rate. However, computing the strain rate using SPH leads to errors at the free surface due to particle deficiency.
Implicit integration scheme

\[ \mathbf{v}(t + \Delta t) = \mathbf{v}^* + \frac{\Delta t}{\rho} \mu \nabla^2 \mathbf{v}(t + \Delta t) \]

Compute Laplacian as

\[ \nabla^2 \mathbf{v}_i = 2(d + 2) \sum_j \frac{m_j}{\rho_j} \frac{\mathbf{v}_{ij} \cdot \mathbf{x}_{ij}}{\|\mathbf{x}_{ij}\|^2 + 0.01h^2} \nabla W_{ij} \]

Solve linear system using a meshless conjugate gradient method.
Laplacian approximation avoids problems at the free surface.
The strain rate based formulation leads to errors and artifacts at the free surface, which is avoided by Weiler et al.

The viscosity parameters of Bender and Peer depend on the temporal and spatial resolution.

Peer’s reconstruction of the velocity field is fast but introduces a significant damping => not suitable for low viscous flow

Takahashi et al. require the second-ring neighbors => low performance
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Quality Comparison

Peer & Teschner 2016

Takahashi et al. 2015

Bender & Koschier 2017

Weiler et al. 2018
Coiling

Peer & Teschner 2016
Bender & Koschier 2017
Takahashi et al. 2015
Weiler et al. 2018
Temperature-Dependent Viscosity
Summary

- **Low viscous flow**
  - Explicit methods are cheap and well-suited
  - Approximation of Laplacian yields better results while XSPH is slightly faster

- **Highly viscous fluids**
  - Implicit methods are recommended to guarantee stability
  - Strain rate based SPH formulations lead to artifacts at the free surface
  - Weiler et al. avoid this problem and generate more realistic results.